Toppersnotes

Unleash the topper in you

CSIR-NET

Council of Scientific & Industrial Research

MATHEMATICAL SCIENCE

VOLUME - II

SIERRA INNOVATIONS

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Set :- A collection of well define and distinct object is define as set.

By well define we mean there is no conclusion regarding the inclusion or exclusion of the objects.

Example:- Collection of vowel's $= \{a, e, i, o, u\}$.

Power set :- A collection of all the subsets of a set is define as the power set.

** If $|A| = n$ then $|P(A)| = 2^n$.

$^nC_0 + {}^nC_1 + \cdots + {}^nC_n = |P(A)|$ —(I)

By Binomial Theorem.

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n$ —(II).

from (I) and (II), we get

$x = 1$

$\therefore (1+1)^n = |P(A)|$

$\Rightarrow |P(A)| = 2^n$.

Prob: If $|A| = 2n+1$ then The number of subset of A having more than n elements are

$|A| = 3$.

$2^3 = 8$

$= 5$

$|A| = 5$,

(I) 2^{n-1} (II) 2^n (III) 2^{n+1}

(IV) 2^{2n} ✓

If $|A| = 2n+1$

$$\underline{{}^{2n+1}C_0 + {}^{2n+1}C_1 + \cdots + {}^{2n+1}C_n} + \underline{{}^{2n+1}C_{n+1} \cdots + {}^{2n+1}C_{2n+1}}$$

$$= |P(A)| = 2^{2n+1}$$

$\because \; {}^nC_r = {}^nC_{n-r}$

${}^{2n+1}C_{n+1} = {}^{2n+1}C_n$

${}^{2n+1}C_{n+2} = {}^{2n+1}C_{n-1}$

$\vdots$

${}^{2n}C_{2n+1} = {}^{2n+1}C_0$

$\alpha + \alpha = 2^{2n+1}$

$2\alpha = 2^{2n+1}$

$\boxed{\alpha = 2^{2n}}$

Cartesian Product:- Let A and B be any any two non-empty sets define $A \times B = \{(a, b) : a \in A, b \in B\}$. then $A \times B$ is define as cartesian product of A and B.

Properties

i) If $|A| = m$, and $|B| = n$ then $|A \times B| = m \cdot n$.

(ii) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

(iii) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

(iv) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Prob :- If A and B have 99 elements common then # elements common to $A \times B$ and $B \times A$ are

(i) 2^{99} (ii) 99^2 (iii) 18 (iv) 100

$$|A \cap B| = 99.$$

$$(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$$

$$|(A \times B) \cap (B \times A)| = |A \cap B| \cdot |B \cap A|$$

$$= 99^2.$$

Relation :-

Let A and B be any two set then any subset of $A \times B$ is define as relation from A to B

Note :- If $|A| = m$, $|B| = n$ then # no. of relation from A to B $= 2^{m \cdot n}$ (# = total).

(ii) Two relation are said to be distinct iff they correspond to different subsets of cartesian product.

(iii) Relation on a set A. i.e. subset of $A \times A$.

Types of relation

Identity Relation :- Let A be any set then any subset of $A \times A$ is define as identity relation if every element of A is related to itself only and it is denoted by I. i.e.

$$I = \{(a, a) : a \in A\}.$$

Example: $A = \{a, b, c\}$.

$I = \{(a,a), (a,b), (b,b), (c,c)\}$.

$I_2 = \{(a,a), (b,b)\}$.

$I_3 = \{(a,a), (b,b), (c,c)\}$ ✓

"Identity relation is always unique."

Reflexive Relation :- Let A be any set let S be subset of $A \times A$. Then S is said to be reflexive if $I \subseteq S$.

Example:- $A = \{a, b, c\}$.

$S_1 = \{(a,a), (a,b), (b,b), (c,c)\}$ ✓

$S_2 = \{(a,a), (b,b), (c,c)\}$ ✓

$S_3 = \{(a,a), (b,b)\}$. ✗

If $|A| = n$ then total number of reflexive relation on $A = 2^{n^2-n}$.

$A = \{a_1, a_2 \ldots a_n\}$. $\quad |A \times A| = n^2$

$$n^2 \left[\begin{array}{ll} n \left\{ \begin{array}{ll} (a_1,a_1) & -1 \\ (a_2,a_2) & -1 \\ \vdots & \\ (a_n,a_n) & -1 \end{array} \right. \\ n^2-n \left\{ \begin{array}{ll} (a_1,a_2) & -2 \\ (a_2,a_1) & -2 \\ \vdots & \\ (a_{n-1},a_n) & -2 \\ (a_n,a_{n-1}) & -2 \end{array} \right. \end{array} \right.$$

$= \underbrace{(1 \times 1 \ldots \times)}_{n \text{ time}} \underbrace{(2 \times 2 \times 2 \ldots 2)}_{n^2-n \text{ (time)}}$.

$= 2^{n^2-n}$.

Irreflexive Relation:- A be any set and $S \subseteq A \times A$, then S is said to be Irreflexive relation if $I \cap S = \phi$.

$A = \{a, b, c\}$.

$S_1 = \{(a,a), (b,b), (c,c)\}$. ✗

$S_2 = \{(a,a), (b,b)\}$. ✗

$S_3 = \{(a,b), (b,c)\}$. ✓

S_2 is neither reflexive nor Irreflexive.

If $|A| = n$, the total number of Irreflexive relation on set $A = 2^{n^2-n}$.

$|A| = n$, $A = \{a_1, a_2, \ldots a_n\}$.

$$n^2 \left\{ \begin{array}{l} n \left\{ \begin{array}{l} (a_1, a_1) \\ \vdots \\ (a_n, a_n) \end{array} \right. \\ n^2-n \left\{ \begin{array}{l} (a_1, a_2) \\ (a_2, a_1) \\ \vdots \\ (a_{n-1}, a_n) \\ (a_n, a_{n-1}) \end{array} \right. \end{array} \right.$$

$$\underbrace{1 \times 1 \times 1 \times 1 - \cdot 1}_{n \text{ time}} \times \underbrace{2 \times 2 \times 2 - \times 2}_{n^2-n \text{ time}}$$

$$= 2^{n^2-n}.$$

Case-I $A = \phi$

$S \subseteq A \times A$

if $S = \phi$ < reflexive / Irreflexive.

Case-II

if $A \neq \phi$

$S = \phi$ < not reflexive / Irreflexive.

Symmetric Relation :-

Let A be any set and $S \subseteq A \times A$ then S is said to be symmetric relation if $(a,b) \in S \Rightarrow (b,a) \in S$

i.e. $a \sim b \Rightarrow b \sim a$.

Example:- $A = \{a, b, c\}$.

$S_1 = \{(a,b), (b,a), (c,c)\}$ ✗

$S_2 = \{(a,a), (a,b), (b,a)\}$ ✓

If $|A| = n$,

$A = \{a_1, a_2, \ldots, a_n\}$.

$$n^2 \begin{cases} n \begin{cases} (a_1, a_1) & -2 \\ (a_2, a_2) & -2 \\ \vdots & \\ (a_n, a_n) & -2 \end{cases} \\ \frac{n^2-n}{2} \begin{cases} \boxed{(a_1, a_2) \\ (a_2, a_1)} & -2 \\ \vdots & \\ \boxed{(a_{n-1}, a_n) \\ (a_n, a_{n-1})} & -2 \end{cases} \end{cases}$$

\# no. of reflexive relation on a set $A = \underbrace{2 \times 2 \cdots 2}_{n \text{ time}} \cdot \underbrace{2 \times 2 \cdots}_{\frac{n^2-n}{2} \text{ time}}$

$= 2^n \cdot 2^{\frac{n^2-n}{2}}$

$= 2^{\frac{n^2+n}{2}}$

$= 2^{\Sigma n}$.

Asymmetric Relation :-

Let A be any set and $S \subseteq A \times A$. Then S is said to be Asymmetric Relation if $(a,b) \in S \Rightarrow (b,a) \notin S$.

Note:- If S is asymmetric $\Rightarrow$ S is Irreflexive.

not irreflexive $\Rightarrow$ asymmetric.

Example. $A = \{a, b, c\}$.

$S_1 = \{(a,a), (a,b)\}$ ✗

$S_2 = \{(a,b), (b,a)\}$ ✗

$S_3 = \{(a,b), (a,c)\}$.

If $|A| = n$.

$A = \{a_1, a_2, \dots, a_n\}$. $\quad |A \times A| = n^2$

$$n \left\{ \begin{array}{l} (a_1, a_1) \longrightarrow 1 \\ (a_2, a_2) \longrightarrow 1 \\ \vdots \\ (a_n, a_n) \longrightarrow 1 \end{array} \right.$$

$$\frac{n^2-n}{2} \left\{ \begin{array}{l} \boxed{\begin{array}{l}(a_1, a_2) \\ (a_2, a_1)\end{array}} \; -3 \\ \vdots \\ \boxed{\begin{array}{l}(a_{n-1}, a_n) \\ (a_n, a_{n-1})\end{array}} \; =3 \end{array} \right.$$

No. of Asymmetric Relation on a set $A = \underbrace{1 \times 1 \times \dots \times}_{n \text{ times}} \underbrace{3 \cdot 3 \cdots 3}_{\frac{n^2-n}{2} \text{ times}}$

$$= 3^{\frac{(n^2-n)}{2}}$$

Anti Symmetric Relation:-

Let A be any set and $S \subseteq A \times A$ then S is said to be anti symmetric if

$(a,b) \in S$, $(b,a) \in S$

$\Rightarrow a = b$.

Example :- $A = \{a, b, c\}$.

$S_1 = \{(a,a), (a,b), (b,a)\}$ ✗

$S_2 = \{(a,a), (a,b)\}$ ✓
(void situation)

if $|A| = n$.

$A = \{a_1, a_2, \dots a_n\}$.

$$n\left\{\begin{array}{l}(a_1, a_1) - 2\\ \vdots \\ (a_n, a_n) - 2\end{array}\right.$$

$$\frac{n^2-n}{2}\left\{\begin{array}{l}\boxed{\begin{array}{l}(a_1, a_2)\\(a_2, a_1)\end{array}} - 3\\ \boxed{\begin{array}{l}(a_1, a_3)\\(a_3, a_1)\end{array}} - 3\\ \vdots \\ \boxed{\begin{array}{l}(a_{n-1}, a_n)\\(a_n, a_{n-1})\end{array}} - 3\end{array}\right.$$

total number of anti symmetric relation on a set $A = \underbrace{2\times 2 \dots 2}_{n\text{-times}} \times \underbrace{3\times 3 \dots 3}_{\frac{n^2-n}{2}}$

$$= 2^n \cdot 3^{\frac{n^2-n}{2}}$$

Problem :- consider $J = \{(a, b) : a, b \in \mathbb{R}$ and $a < b\}$.

Let (a, b) and $(c, d) \in J$

Define R on J as $(a, b) R (c, d) \Leftrightarrow a \le c$ and $d \le b$.

check wheather R is antisymmetric or not.

solution:- $J = \{(1, 2), (2, 3), (3, 4), \dots\}$. X

$(a, b) R (c, d)$ and $(c, d) R (a, b)$.

$a \le c$ and $b \le d$ $\qquad$ $c \le a$, & $d \le b$

$a \le c$ and $b \le d$
$c \le a$ $\qquad$ $d \le b$

$\Rightarrow a = b$ and $b = d$

Transitive Relation :-

Let A be any set and $S \subseteq A \times A$ then S is said to be transitive relation if $(a, b) \in S$, $(b, c) \in S$

$\Rightarrow (a, c) \in S$.

Equivalence Relation :-

A Relation is said to be equivalence relation if it is reflexive, symmetric, and transitive.

Problem:- which of the following relation define on the set of integer ~~set~~ is an equivalence relation.

a) $m \sim n \iff m \cdot n \geq 0$ — $-2 \sim 0$, $0 \sim 2$ but $-2 \not\sim 2$,

b) $m \sim n \iff m \cdot n > 0$ — reflexive not

c) $m \sim n \iff |m| = |n|$ ✓

d) $m \sim n \iff mn \leq 0$. — reflexive not.

(c) $(a, b) \sim a \sim b \Rightarrow a \cdot b \geq 0$,

$\Rightarrow$ reflexive.

$(1, 4)$ $(4, 2)$ ✗

$0, 1)$ $0 \sim$

Problem: Let $A = \mathbb{R}$, Let $x, y \in \mathbb{R}$ Define $\sim$ on A. as $x \sim y \iff y \leq x^2$ Then $\sim$ is

(I) reflexive (II) symmetric (III) transitive

(IV) None of These ✓

not reflexive $\leftarrow$ $-1 \not\sim -1$ — not symmetric

$2 \sim 2$ but $1 \not\sim 2$ —

$-\frac{3}{2} \sim 2$ and $2 \sim 3$

but $-\frac{3}{2} \not\sim 3$

$\Rightarrow$ Not transitive.

Hence option is N.O.T.

Partition of a set:-

Let A be any non-empty set

Let $P = \{ A_i : i \in I$, where I is non-empty index set $\} \subseteq P(A)$

where $A_i \subseteq A$ and $A_i \neq \phi$

satisfying.

if $x \neq y$ and $x, y \in P \Rightarrow x \cap y = \phi$

$\bigcup_{i \in I} A_i = A$.

Then P is defined as partition of A.

Note:- If A is empty set then partition of A = $\{\phi\}$."

Example:- $A = \{ a, b, c \}$.

$A_1 = \{a\}$, $A_2 = \{b, c\}$.

$P_1 = \{A_1, A_2\}$.

P_1 is partition of A.

$A_1 = \{b\}$, $A_2 = \{a, c\}$.

$P_2 = \{A_1, A_2\}$.

P_2 is partition on A.

$A_1 = \{c\}$, $A_2 = \{a, b\}$.

$P_3 = \{A_1, A_2\}$.

P_3 is Partition on A.

$A_1 = \{a\}$. $A_2 = \{b\}$. $A_3 = \{c\}$.

$P_4 = \{A_1, A_2, A_3\}$.

P_4 is Partition on A.

$A_1 = \{a, b, c\}$.

$P_5 = \{A_1\}$.

Problem :- Let A be the set of Real numbers. define
$P_1 = \{[n, n+1] \ . \ n \in \mathbb{Z}\}$.
$P_2 = \{(n-1, n+1) \ : n \in \mathbb{Z}\}$.

Then.

i) Both are partition of A.
ii) None of them are partition of A.
iii) P_1 is partition of A but not P_2
iv) P_2 is " " "

$A_1 = [0, 1], A_2 [1, 2] \ldots\}$. $A_1 \neq \phi$, $A_2 \neq 0 \Rightarrow A_1 \cap A_2 \neq \phi$.

P_1 is ~~not~~ Partition.

$P_2 = \{ (0, 2), (1, 3), \ldots \}$

$A_1 = (0, 2)$, $A_2 = (1, 3)$.

$\Rightarrow A_1 \cap A_2 \neq \phi$.

Hence P_2 is not Partition.

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Note :-

Let $P(n)$ denotes no. of partition of a set whose cardinality is 'n'.

$P(0) = 1$

$P(1) = 1$

$P(2) = 2$

$P(3) = 5$

$$\boxed{P_{n+1} = \sum_{r=0}^{n} {}^{n}C_r \, P(r)}$$

$$P_{3+1} = \sum_{r=0}^{3} {}^{3}C_r \, P(r)$$

$$= {}^3C_0 P(0) + {}^3C_1 P_1 + {}^3C_2 P_2 + {}^3C_3 P_3$$

$$= \frac{\lfloor 3}{\lfloor 0 \lfloor 3} P(0) + \frac{\lfloor 3}{\lfloor 1 \lfloor 2} \cdot 1 + \frac{\lfloor 3}{\lfloor 2 \lfloor 1} \cdot 2 + \frac{\lfloor 3}{\lfloor 3 \lfloor 0} \cdot 5$$

$$= 1 + 3 + 6 + 5 = 15.$$

Equivalence class :-

Let A be non-empty set and R be any equivalence relation on A. Let $a \in A$. Then equivalence class of a is defined as set of those elements of A which are related to a by the equivalence relation R and it is denoted by $cl(a)$.

$$cl(a) = \{ x \in A : xRa \} \subseteq A.$$

properties of an Equivalence class :-

i) $a \in cl(a)$ i.e. $cl(a) \neq \phi$

ii) Either $cl(a) \cap cl(b) = \phi$ or $cl(a) = cl(b)$

iii) $aRb \Leftrightarrow cl(a) = cl(b)$

(iv) $A = \bigcup_{a \in A} cl(a)$

quotient set:- collection of all distinct equivalence classes on a set A by some equivalence relation R. is define as the quitent set and it is denoted by A|R. i.e.

$$A|R = \{ cl(a) : a \in A \} \subset P(A).$$

Note:- A|R define a partition on a set A.

Example:- $A = \mathbb{R}$

let $a, b \in \mathbb{R}$

Define $\sim$ on $\mathbb{R}$ as

$a \sim b \iff [a] = [b]$

clearly $\sim$ is an equivalence relation

$\therefore$ It partitions set A.

$\vdots$

$cl(-2) = [-2, -1)$

$cl(-1) = [-1, 0)$

$cl(0) = [0, 1)$

$cl(1) = [1, 2)$

$cl(2) = [2, 3)$

$\vdots$

$A|\sim = \{ cl(a) : a \in \mathbb{R} \}.$

$= \{ \ldots\ldots [-1, 0), [0, 1), [1, 2), \ldots\ldots \}.$

$|A|\sim| = \aleph_0.$

Example :-

$A = \mathbb{R}$

Let $a, b \in \mathbb{R}$

Define $\sim$ on $\mathbb{R}$ as

$a \sim b \iff$ either both are rational or both are irrational.

Clearly, $\sim$ is an equivalence relation

$\therefore$ It partition set A.

Let $a \in \mathbb{Q}$.

$cl(a) = \{x \in \mathbb{R} : x \sim a\}$.

$= \mathbb{Q}$

Let $a \in \mathbb{Q}^c$

$cl(a) = \mathbb{Q}^c$.

$A/\sim = \{\mathbb{Q}, \mathbb{Q}^c\}$.

$|A/\sim| = 2$.

Example :-

$A = \mathbb{R}$

Let $a, b \in \mathbb{R}$ as

$a \sim b \iff a - b \in \mathbb{Z}$

Clearly, $\sim$ is an equivalence relation

$\therefore$ It partitions set A.

Let $a \in \mathbb{R}$.

$cl(a) = \{x \in \mathbb{R} : x \sim a\}$.

$= \{x \in \mathbb{R} : x - a \in \mathbb{Z}\}$.

$= \{x \in \mathbb{R} : x - a = n \in \mathbb{Z}\}$.

$= \{ a+n : n \in \mathbb{Z} \}$.

$cl(a)$ is countable $\forall a \in \mathbb{R}$.

$A/\sim = \{ cl(a) : a \in A \}$.

Suppose $A/\sim$ is countable

$A/\sim = \{ cl(a_1), cl(a_2), \ldots \}$.

$\bigcup_{i \in \mathbb{N}} cl(a_i) = \mathbb{R}$.

countable = uncountable

$\times$

$\therefore$ $A/\sim$ is uncountable.

Note :- Set is uncountable then its quotient set can be countably infinite, finite or uncountable.

ii. Set is countable then its quotient set is always countable.

Fundamental Theorem of an equivalence ~~Relation~~ relation :-

Every equivalence relation defines a partition and conversely.