Toppersnotes
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CSIR-NET

Council of Scientific & Industrial Research

MATHEMATICAL SCIENCE

VOLUME - VII

SIERRA INNOVATIONS

INDEX

Sets and its Fundamentals

Sets – A well define collection of distinct objects is called set.

Note: By well define we mean there is no ambiguity confusion regarding inclusion or exclusion of any object.

2. collection / set builder notation

$$X = \{ \underset{\text{Type of object}}{\underline{\qquad}} : \underset{\text{Rule for collection}}{\underline{\qquad}} \}.$$

3. Generally sets are denoted by capital letter A, B... etc. Where as objects included in the set, called elements, denoted by small letter a, b, c... etc.

4. **NAC:** When we define a set, the set itself became a object and elligible to be collected for some other set.

   If x is an object collected in the set X, we write $x \in X$. (x belongs to X).

Axioms of Regularity:
If X is a set then $X \notin X$

Note: Embty collection of well define has be a set without object called embty set, Void set or null set, it is denoted by $\{\}$ or $\emptyset$

(i) ordinary set: x is ordinary set if $x \notin x$.

(ii) extra ordinary set : $x \in x$.

Russel's Paradox:-

There is no set of all set.

$X = \{ A$ is set; $A \notin A \}$.

= The collection of ordinary set.

Axioms of choice :-
If we have collection of non-empty set, then we can choose one element from each set

Subset :- Let A and B are set s.t.

if $x \in A$
$\Rightarrow x \in B$
Then we say A is a subset of B and we write $A \subset B$

if $A \subset B$ and $\exists x \in B$ s.t. $x \notin A$
$A \subsetneq B$

If $\nexists x \in A$ s.t. $x \notin B$.
then we say A is subset of B, and we write.
$A \subset B$

$\nexists x \in \{ \}$ s.t. $x \notin X$ (any).

$\{ \}$ or ϕ

$\Rightarrow$ Empty set is subset of every set

A and A are set

$x \in A \Rightarrow x \in A$

$\Rightarrow A \subset A$

Every set is a subset of itself.

<u>Power set</u> :- Let X be a set Then

$P(X) = \{A : A \subset X\}$.

= The set of all the subset of X

called the Power set.

$\Rightarrow$ for any set X, <u>$P(X)$ is never empty</u>.

$P(\phi) = \{\phi\}$.

<u>Note</u> :- $X = \{a, b, c, \{a, b\}, \}$.

$\alpha = \{a, b\}$.

<u>Cartesian Product</u> :-

Let A and B are set

$a \in A,\ b \in B$

$\alpha_{a,b} = \{a, \{a, b\}\} \subset A \cup P(A \cup B)$

$\alpha_{c,d} = \{c, \{c, d\}\}$.

if $\alpha_{a,b} = \alpha_{c,d}$

$\Rightarrow \{a, \{a, b\}\} = \{c, \{c, d\}\}$.

$\Rightarrow a = c,\ b = d$

Adopt a notation

$(a, b) = \alpha_{a,b}$

$(c, d) = \alpha_{c,d}$

$(a,b) = (b,a)$

$\Rightarrow a = b$.

$$\boxed{A \times B = \{ (a,b) : a \in A, b \in B \}.}$$

similarly

$A \times B \times C = \{ (a,b,c) : a \in A, b \in B, c \in C \}$.

$\mathbb{N} = \{ 1, 2, 3, \ldots\ldots n, n+1 \ldots\ldots \}$

= The set of natural symble.

$S(n)$ = An initial segment of $\mathbb{N}$.

Symbole: board:-

Function :- Let A and B are non empty set then a rule by which every element of A is assign to some unique element of B define a function from A to B and denoted by $f: A \rightarrow B$.

if $x \in A$ is assigned to $y \in B$.

we write $y = f(x)$.

y is called The image of x under f and

x is called 'a' Pre image.

A : domain.

B : codomain.

$f(A) = \{ f(x) : x \in A \} \subset B$. called The range of 'f'

Types:

one-one function:

$$f(a) = f(b)$$
$$\Rightarrow a = b$$

onto or surjection function:

$$f(A) = B.$$

Bijection : 1-1 and onto both.

Similar set :- Two non-empty sets are said to be similar if there exist a bijection map between them. The terms like equivalent, or equinumerous, or equipotential. are also used in place of similar

Note:- If $f: A \to B$ onto fails to exist (there is no function onto.) we say B is more potential than A.

$$|A| \leq |B|$$

2. If there exist one-one function from A to B then B has more or equal potential than A.

$$|A| \leq |B|$$

Cantor's Theorem:

from any set to itse' power set cannot be defined onto function

Hence the potential of power set has always more than set itself.

Let $f : A \xrightarrow{\text{onto}} P(A)$

$x \in A \Rightarrow f(x) \in P(A)$

$\Rightarrow f(x) \subset A$

then for any $x \in A$

either $x \in f(x)$ or $x \notin f(x)$

$X = \{ x \in A : x \notin f(x) \}$.

$\Rightarrow X \subset A \Rightarrow X \in P(A)$

as f is onto

$\exists \alpha \in A$

s.t. $f(\alpha) = X$

$\alpha \in X$	$\alpha \notin X$
$\alpha \notin f(\alpha)$	$\alpha \in f(\alpha)$
$\alpha \notin X$	$\alpha \in X$

$\Rightarrow f$ can not be onto.

Finite set :- if $A \sim S_n$ for some $n \in \mathbb{N}$.

$\Rightarrow A$ is finite, n is called the cardinality of A. and denoted by $|A|$,

& $|A| = n$.

Infinite :- if $A \neq \phi$ and $A \nsim S_n$ for any $n \in \mathbb{N}$.

$\Rightarrow A$ is infinite

Example - $\mathbb{N}$.

$|\mathbb{N}| = \aleph_0$ (aleph Naught)

$|P(\mathbb{N})| = c$ (continuum)

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<u>continum Hypothesis</u> :-

There is no transfinite number (symbole) between aleph nought $\aleph_0$ and continum 'c' i.e. there is no set whose potential is more than $\mathbb{N}$ and less than $P(\mathbb{N})$. This is know as continum Hypothesis.

<u>NAC</u> :- The element of any set can be put under by counting processes if there is rule by which first member, second member - - - etc are define and sucessor of every member of the set is define when the set gets exausted.

Example: The element S_n and $\mathbb{N}$ are naturally under counting process.

<u>countable set</u> :- A set is said to be countable if it is either empty or finite or similar to $\mathbb{N}$.

<u>countabily infinite</u> :- A set is similar to $\mathbb{N}$, is called countably infinite moreover if A is countablity infinite

i.e. $A \sim \mathbb{N}$ then.

we can write $A = \{a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots\}$.

Note :- A set is countable if it is finite or countably infinite.

(ii) Every countably infinite set has cardinality $\aleph_0$.

Uncountable set : A set is uncountable if it is neither finite nor countably infinite. i.e. not countable.

Example:- $P(\mathbb{N})$, $P(P(\mathbb{N}))$, $P(P(P(\mathbb{N})))$

Lemma I: Every infinite set has a countably infinite subset. Hence $\aleph_0$ is the smallest transfinite number.

Proof :- Let X is an infinite set.

by AOC, select $x \in X$ and name it $x = a_1$

define $A_1 = \{a_1\}$.

now again select $x \in X - A_1$ and name it $x = a_2$

define $A_2 = A_1 \cup \{a_1\}$
$= \{a_1, a_2\}$.

Similarly $A_3 = \{a_1, a_2, a_3\}$.

$\Rightarrow$ A_n is defined $\forall n \in \mathbb{N}$.

define $A = \bigcup_{n \in \mathbb{N}} A_n$

$\Rightarrow A = \{a_1, a_2, a_3, \ldots a_n, a_{n+1}, \ldots\}$

A is countably infinite and $A \subset X$.

Now if $f : \mathbb{N} \to X$
as $f(r) = a_r$ $\Rightarrow f$ is 1-1

$\Rightarrow \quad |\mathbb{N}| \leq |X|$

$\aleph_0 \leq |X|$

Lemma-II :- Every infinite set is similar to at least one of its proper subset more over if X is infinite set $F \subset X$, F is finite, $X \sim X-F$.

Proof :-

X is infinite

$\exists A \subset X$.

$A = \{a_1, a_2, \ldots, a_n, \ldots\}$

$X = (X-A) \cup A$.

$B = A - \{a_1\}$.

$Y = (X-A) \cup B$.

$\Rightarrow \quad Y \subsetneq X$ as $a_1 \notin X$.

$f: X = (X-A) \cup A \longrightarrow Y = (X-A) \cup B$.

$$f(x) = \begin{cases} x & x \in X-A \\ a_{i+1} & x = a_i \end{cases}$$

f is 1-1 and onto

$X \sim Y$.

Note :- If every proper subset of a set is countable then set has to be countable. (Proof Lemma-2).

12. If every countable subset of S is a proper subset of S then S is uncountable set

Example P(N). since every proper subset of P(N) is countable but P(N) is uncountable.

Some Important similar sets:-

#1. for any a, b, $(a \neq b) \in \mathbb{R}$

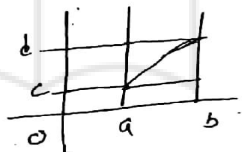
$[a,b] \sim (a,b) \sim [a,b), \sim (a,b]$ (by lemma 2).

#2. Let $I = [a,b]$, $a \neq b$

$J = [c,d]$ $c \neq d$.

define $f : I \longrightarrow J$

$f(x) = \dfrac{d-c}{b-a}(x-a)+c$



f is 1-1 and onto.

$\Rightarrow I \sim J.$ $\Rightarrow$ $[a,b] \sim [c,d]$

$[a,b] \sim [c,d]$
$\sim [c,d)$
$\sim [c,d]$

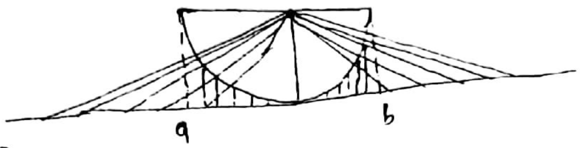
i.e. Any two non-trivial intervals also are similar set.

#3. $f : (-\pi/2, \pi/2) \longrightarrow \mathbb{R}.$

$f(x) = \tan x.$

$(a,b) \sim (a,b) \sim (-\pi/2, \frac{\pi}{2}) \sim \mathbb{R}$
$\sim [a,b] \sim \mathbb{R}$
$\sim [a,b) \sim \mathbb{R}.$

#4.



$A = (a, b)$

= The set of points on the semi circle

$C = \mathbb{R} = (-\infty, \infty)$

$A \sim B \sim C = \mathbb{R}$

$\Rightarrow (a, b) \sim \mathbb{R}$

$(c, d) \sim \mathbb{R}$

$(a, b) \sim (c, d)$

#5. Let A and B are non-empty set

$B^A = \{ f \mid f : A \to B \}$.

a new Notation

then $|B^A| = |B|^{|A|}$

Example: $\mathbb{N}^{\mathbb{N}} = \{ f \mid f : \mathbb{N} \to \mathbb{N} \}$.

$\boxed{|\mathbb{N}^{\mathbb{N}}| = \aleph_0^{\aleph_0}}$

Similarly.

$|\{0, 1\}^{\mathbb{N}}| = 2^{\aleph_0}$

\# If $X \subset A \neq \phi$

$\exists\ f_X : A \to \{0,1\}$.

$$f_X(x) = \begin{cases} 1 & x \in X \\ 0 & x \notin X. \end{cases}$$

This is known as characteristic function.

if $Y \subset A \Rightarrow \exists\ f_Y : A \longrightarrow \{0,1\}$.

$$f_Y(x) = \begin{cases} 1 & x \in Y \\ 0 & x \notin Y \end{cases}$$

if $X \neq Y$

$\Rightarrow f_X \neq f_Y$

\# If $X \in P(A)$

$\Rightarrow X \subset A$

$\exists\ f_X : A \longrightarrow \{0,1\}^A$

$$f_X(x) = \begin{cases} 1 & x \in X \\ 0 & x \notin X \end{cases}$$

$\Rightarrow f_X \in \{0,1\}^A$

define : $\phi : P(A) \longrightarrow \{0,1\}^A$.

$\phi(X) = f_X$

if $\phi(X) = \phi(Y)$

$\Rightarrow f_X = f_Y$

$\Rightarrow X = Y$.

ϕ is 1-1.

$g \in \{0,1\}^A$

$\Rightarrow g: A \to \{0,1\}$

$T = \{x \in A : g(x) = 1\}.$

$\Rightarrow T \subset A$

$\Rightarrow T \in P(A)$

$f_T : A \to \{0,1\}.$

$$f_T(x) = \begin{cases} 1 & x \in T \\ 0 & x \notin T. \end{cases}$$

$f_T(x) \in \{0,1\}^A. \Rightarrow f_T = g.$

$\phi(T) = f_T = g.$

ϕ is onto

Imp

$P(A) \sim \{0,1\}^A$

in particular

$P(\mathbb{N}) \sim \{0,1\}^{\mathbb{N}}.$

$\boxed{c = 2^{\aleph_0}}$

$\{0,1\}^{\mathbb{N}} \sim P(\mathbb{N})$

$\Rightarrow \{0,1\}^{\mathbb{N}}$ is uncountable

and $|\{0,1\}^{\mathbb{N}}| = c.$

Now define $S = \{\langle a_n \rangle : a_n = 0,1\}.$

$S \sim \{0,1\}^{\mathbb{N}} \sim P(\mathbb{N})\}.$

The set of all the sequence of 0 and 1 is uncountable with cardinality c.

\# if $P(A)$ is infinite.

$\Rightarrow A$ is infinite.

$\Rightarrow |A| < |P(A)|$

$\Rightarrow P(A)$ cannot be countably infinite

i.e. $|P(A)| \neq \aleph_0$ for any A.

Tools For countability

T_1: If there exist one-one function from A to $\mathbb{N}$. (any countable set) then A is countable.

T_2: Let X be countable set and $A \subseteq X$

define $f: A \to X$

$f(x) = x$.

$\Rightarrow f$ is one-one.

$\Rightarrow$ by T_1, A is countable.

"Every subset of countable set is countable."

T_3:- if A and B are countable then $A \times B$ is countable.

Proof:- first we claim $\mathbb{N} \times \mathbb{N}$ is countable.

$f: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

$f(m,n) = 2^m \cdot 3^n$

$f(m,n) = f(r,s)$

$2^m \cdot 3^n = 2^r \cdot 3^s$

$\Rightarrow 2^{m-r} = 3^{s-n}$

$\Rightarrow m-r = 0, \quad s-r=0.$

$m = r$ and $s = n$

$\Rightarrow (m, n) = (r, s)$.

f is 1-1

by T1, $\mathbb{N} \times \mathbb{N}$ is countable.

Now let

$A = \{ a_1, a_2, \ldots \ldots a_n, a_{n+1}, \ldots \}$

$B = \{ b_1, b_2, \ldots \ldots b_n, b_{n+1}, \ldots \}$.

$A \times B = \{ (a_i, b_j) : a_i \in A, b_j \in B \}$.

define $\phi : A \times B \longrightarrow \mathbb{N} \times \mathbb{N}$

$\phi(a_i, b_j) = (i, j)$

$\Rightarrow \phi$ is one-one and onto

$A \times B \sim \mathbb{N} \times \mathbb{N} \sim \mathbb{N}$.

$\Rightarrow A \times B$ is countable set

T4 : If $A_1, A_2, \ldots \ldots A_k, A_{k+1} \ldots$ are countable sets then for any $n \in \mathbb{N}$, then

$$A_1 \times A_2 \times \ldots \times A_m = \prod_{i=1}^{m} A_i \text{ is countable.}$$

gt is finite cartesian product of countable set is countable.

Example:- $\mathbb{Z}$ is countable.

$\mathbb{Z} \times \mathbb{Z}$ is countable.

Now write $p/q \in \mathbb{Q}$.

as $p/q = (p, q) \in \mathbb{Z} \times \mathbb{Z}$