Toppersnotes

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CSIR-NET

Council of Scientific & Industrial Research

MATHEMATICAL SCIENCE

VOLUME - IV

SIERRA INNOVATIONS

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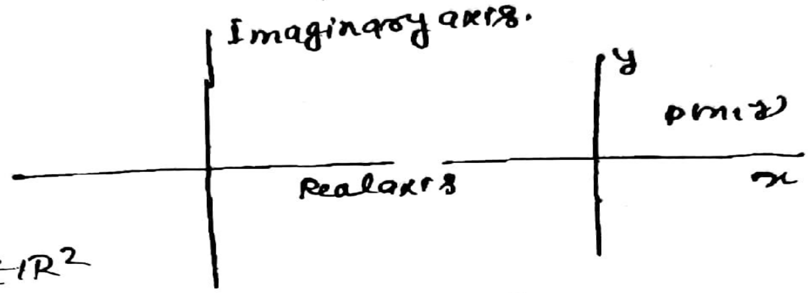
complex number & its fundamental.

$\mathbb{C} = \{x+iy : x, y \in \mathbb{R}\}$

$\mathbb{C}$ is a field with respect to ordinary + and ·



$z \in \mathbb{C}$

$z = x + iy$

$z \sim (x, y) \in \mathbb{R}^2$

for each $z \in \mathbb{C}$ $\exists$ unique $(x, y) \in \mathbb{R}^2$

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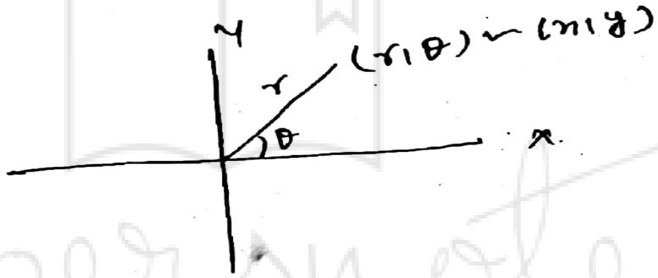


$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

$\tan\theta = \frac{y}{x}$

$z = x + iy$

$= r\cos\theta + i\sin\theta = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}, \quad r \neq 0$

$e^{i\theta} = \cos\theta + i\sin\theta$

Note :- where $z = re^{i\theta}, \quad r \neq 0$

$\Rightarrow z \neq 0$

$z = re^{i\theta}$

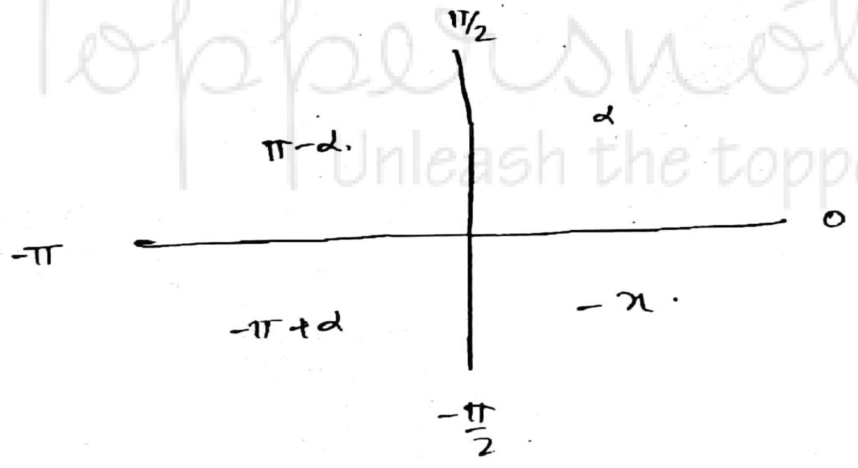
$|z| = r$

$\operatorname{Arg} z = \theta$ where θ is given by.

First find $\boxed{\alpha = \tan^{-1}\left|\frac{y}{x}\right|} \Rightarrow 0 \le \alpha \le \pi/2$

Arg z = Principle Argument

$$\theta = \begin{cases} 0, & x>0,\ y=0 \\ \alpha & x>0,\ y>0 \\ \pi/2 & x=0\ \ y>0 \\ \pi-\alpha & x<0,\ y>0 \\ \pi & x<0,\ y=0 \\ -\pi+\alpha & x<0,\ y<0 \\ -\frac{\pi}{2} & x=0,\ y<0 \\ -\alpha & x>0,\ y<0 \end{cases}$$

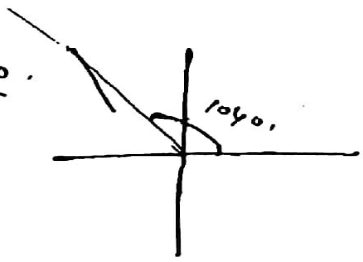


Example –

$z = -1 + 4i$

$\alpha = \tan^{-1}(y/x) = \tan^{-1}\left|\frac{4}{-1}\right| = 76^\circ$

$\theta = \pi - 76^\circ = 104^\circ$.



For a position.

$$\arg z = \{ 2n\pi + \operatorname{Arg} z : n \in \mathbb{Z} \}$$

$$z = x + iy, \quad z = re^{i\theta}.$$

<u>Exponents | complex variable:-</u>

$$a^z, \quad a < x\,?$$

$$a^{x+iy} = a^x a^{iy}$$

$$= a^x \left(e^{\log a}\right)^{iy}.$$

$$= a^x \; e^{i(y \log a)}$$

if $z \in \mathbb{C} - \{0\}$

$$\operatorname{Log}_e z = \log|z| + i \operatorname{Arg} z.$$

$$\operatorname{Log}_e z = \text{p.v of } \log z$$

$$= \text{p.v.} \{ \log|z| + i(\operatorname{Arg} z + 2n\pi) \}$$

<u>Example:-</u> p.v. $\log(1-i)$

$$= \log(1-i)$$

$$= \log\sqrt{2} - i\frac{\pi}{4}.$$

$$\text{p.v of } \log(-1) = \log|-1| + i\pi$$

$$= \log 1 + i\pi$$

$$= i\pi.$$

Let $z_1, z_2 \in \mathbb{C}$

$z_1 \neq 0, \quad z_2 \neq 1.$

$$\text{P.V } z_1^{z_2} = e^{z_2 \text{ P.V of } \log z_1}$$

$$\boxed{\text{P.V } z_1^{z_2} = e^{z_2(\log|z_1| + i \operatorname{Arg} z_1)}}$$

Example:- Find the P.V of $i^{(1+i)}$

$$\text{P.V } i^{(1+i)} = e^{(1+i)(\log|i| + i\frac{\pi}{2})}$$

$$= e^{(1+i)(i\pi/2)}$$

$$= e^{i\pi/2 - \pi/2}$$

$$= e^{i\pi/2} e^{-\pi/2}$$

$$= (\cos\pi/2 + i\sin\pi/2)\, e^{-\pi/2}$$

$$= i e^{-\pi/2}$$

$$\text{Re P.V } i^{(1+i)} = 0$$

$$\text{Img P.V } i^{(1+i)} = e^{-\pi/2}$$

Example:- Rc P.V. of $(1+i)^{(1-i)}$

$$= e^{(1-i) \text{ P.V. of } (1+i)}$$

$$= e^{(1-i)\operatorname{Log}_e(1+i)}$$

$$= e^{(1-i)(\log\sqrt{2} + i\pi/4)}$$

$$= e^{\log\sqrt{2} + \pi/4}\, e^{(-i\log\sqrt{2} + \pi/4)}$$

$$= e^{\log\sqrt{2} + \pi/4}\, e^{i(\frac{\pi}{4} - \log\sqrt{2})}$$

$\sqrt{2}\, e^{\pi/4} (\cos d + i \sin d)$

$= \sqrt{2}\, e^{\pi/4} \cos d + i \sqrt{2}\, e^{\pi/4} \sin d.$

Re. P.V of $(1+i)^{(1-i)} = \sqrt{2}\, e^{\pi/4} \cos(\frac{\pi}{4} - \log \sqrt{2})$

Example:. $(i^{\circ})^{i^{\circ}} = e^{i \log i^{\circ}}$

$= e^{i} (\log|i| + i \pi/2)$

$= e^{i(i\pi/2)} = e^{-\pi/2}$

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Sterioographic projection:-

Let $x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{4}$ ——— (I) be an sphere in $\mathbb{R}^3$

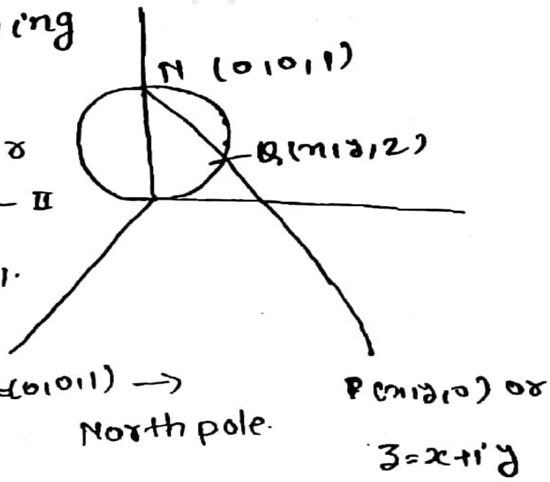
$S = \{ (x,y,z) : x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{4} \}.$

$S_1 = \{ S - (0,0,1) \}.$

$S_2 = C = \{ (x,y,0) : x,y \in \mathbb{R} \}.$

Equation of the line joining NP :—

$$\frac{X-0}{x-0} = \frac{Y-0}{y-0} = \frac{Z-1}{0-1} = r \quad \text{— II}$$



$X = xr,\ Y = yr,\ Z = -r+1.$

Putting the value S in (I)

$x^2r^2 + y^2r^2 + (-r+1-\frac{1}{2})^2 = \frac{1}{4}$

$x^2r^2 + y^2r^2 + r^2 + \frac{1}{4} - r = \frac{1}{4}$

$r^2(x^2+y^2+1) = r.$

$1+x^2+y^2$

$$\Rightarrow \boxed{X = \frac{x}{1+x^2+y^2} \qquad Y = \frac{y}{1+x^2+y^2} \qquad Z = \frac{x^2+y^2}{1+x^2+y^2}} \quad -1$$

$x = \frac{X}{r}, \quad y = \frac{Y}{r}, \quad r = 1-Z$

$$\boxed{x = \frac{X}{1-Z}, \quad y = \frac{Y}{1-Z}} \quad — \text{II}$$

$$z = x+iy = \frac{X}{1-Z} + i\frac{Y}{1-Z}$$

Equation 1 & and II provides one-one onto correspondence between the points in the xy-plane and points on the sphere without north pole.

These fours equation together called stereographic projection of the complex plane to the sphere and conversly.

Chordal distance:-

$$\chi(z_1, z_2) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$$

if $z_1 = (x_1, y_1, z_1)$

$z_2 = (x_2, y_2, z_2)$.

chordal distance b/ $(1+i, \infty)$

$\chi(1+i, \infty) = \sqrt{}$

$x = \frac{1}{3}$ $\quad z_1 = (1/3, 1/3, 2/3)$

$y = 1/3$ $\quad z_2 = (0, 0, 1)$

$z = 2/3$

$$\chi(1+i, \infty) = \sqrt{\left(\tfrac{1}{3}-0\right)^2 + \left(\tfrac{1}{3}-0\right)^2 + \left(\tfrac{2}{3}-1\right)}$$

$$= \sqrt{\tfrac{1}{9} + \tfrac{1}{9} + \tfrac{1}{9}}$$

$$= \sqrt{\tfrac{3}{9}} = \frac{1}{\sqrt{3}} \text{ Ans.}$$

Let us define ∞ and extended complex number $= \mathbb{C} \cup \{\infty\}$.

where $\infty \longrightarrow (0,0,1)$

This ∞ behaves as a number but where extended complex plane with following arithematic -

(i) $z + \infty = \infty, \ \forall z$

(ii) $\infty - z = \infty + (-z) = \infty$

(iii) $\frac{3}{0} = \infty$

(iv) $\frac{3}{\infty} = 0 \quad \forall z \in \mathbb{C}$

If the chordal distance b/w any two points z_1 and z_2 is 1. then these points are called antipodal distance.

i.e. $\chi(z_1, z_2) = 1$

$\Rightarrow$ z_1 and z_2 are called antipodal points

If z_1 and z_2 are corresponding to diametrically oppositive point on the sphere called antipodal point.

Function :-

$f : D \longrightarrow \mathbb{C}$

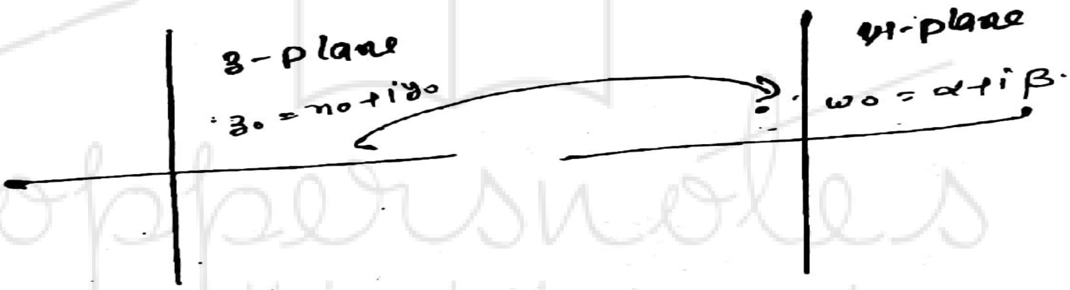
$D \subseteq \mathbb{C}$

we write $f(z) = w$

f is called function of complex variable.

Note :-

1. A function of complex variable corresponds between two copies of complex plane. one is reffered as z-plane and other w-plane.



2. As $z = x + iy$

$w = f(z)$ can be written as

$w = f(z) = u(x,y) + i\,v(x,y)$.

where u and $v : S \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$.

$S = \{ (x,y) : x + iy \in D \}$

Example $f(z) = \frac{1}{z} = z^{-1}$

$D = \mathbb{C} - \{0\}$.

$f(z) = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$

$= \frac{x}{x^2+y^2} + i \frac{(-y)}{x^2+y^2} = u + iv$

3. If ϕ_1 and ϕ_2 are real valued function of two variables defined on $S \subseteq \mathbb{R}^2$ and D analogus to S.

then

$$f : D \longrightarrow \mathbb{C}$$

such that

$$f(z) = f(x+iy) = \phi_1(x,y) + i\,\phi_2(x,y).$$

defines a function of complex variable.

if $x_0 + iy_0 \in D$

$$\Leftrightarrow (x_0, y_0) \in S$$

$\phi_1(x_0, y_0) = \alpha$, $\quad \phi_2(x_0, y_0) = \beta$.

$$\boxed{f(z_0) = \alpha + i\beta}$$

4. if ϕ_1 and $\phi_2 \in \mathbb{R}[x,y]$

then $f(z) = \phi_1(x,y) + i\phi_2(x,y)$ is a well defined function on $\mathbb{C}$.

5. we have

$z = x+iy$, $\quad \bar{z} = x - iy$.

$x = \dfrac{z+\bar{z}}{2}$ $\quad y = \dfrac{z-\bar{z}}{2i}$

$\Rightarrow z(x,y) = z$ & $\bar{z}$ $(= \bar{z}(x,y))$ $\quad \Big| \quad x = r\cos\theta$

$x = x(z,\bar{z})$, $\quad y = y(z,\bar{z})$ $\quad \Big| \quad y = r\sin\theta$

$\Rightarrow$ $f(z) = f(z, \bar{z})$ $\qquad\qquad f(z) = f(r,\theta)$.

$= f(x,y)$

$= f(\bar{z})$

$= f(z)$

$= f(x)$ and $f(y)$

Let $w = f(z) = u(x,y) + i v(x,y)$.

we have z- equation / inequation in x, y its represents curve and region in z-plane. and if with the help the equation / inequation we fined equation / inequation then represents image curve / image in u and v. Region.

Example:-

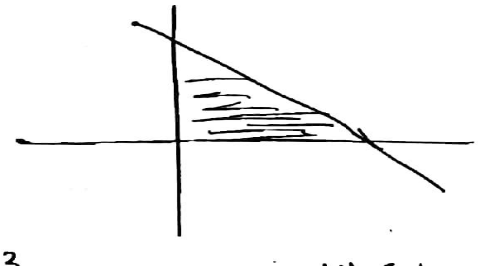
$$f(z) = 2z + 3 + 2i$$

$$= 2(x+iy) + 3 + 2i$$

$$= (2x+3) + (2y+2)i$$

$u = 2x+3, \quad v = 2y+2$

$x+y \le 1, \quad x \geq 0, \quad y \geq 0$



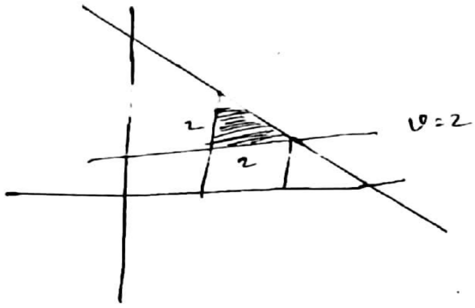
$\frac{u-3}{2} = x, \qquad x+y \le 1$

$\frac{v-2}{2} = y \qquad \frac{u-3}{2} + \frac{v-2}{2} \le 1$

$u + v \le 3$

$u \geq 3$

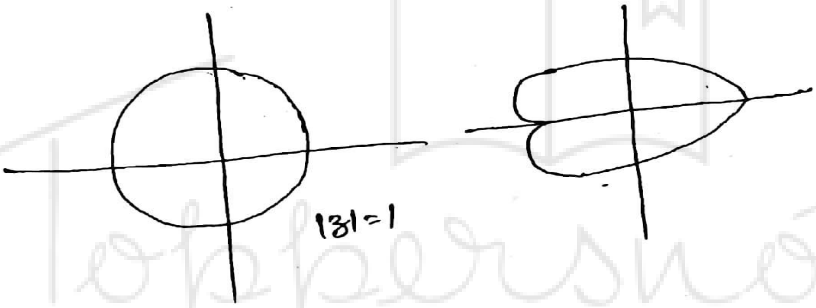
$\frac{v-2}{2} \geq 0 \Rightarrow v \geq 2$



$\Delta A_z = \frac{1}{2}$

$\Delta A_w = 2. = 4 \cdot \Delta A_z.$

Example:- $f(z) = z^2 + 2z.$



some Important function

$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots \quad + a_n z^n \Rightarrow P_1(x,y) + i P_2(x,y)$

$P(x,y) \in R(x,y)$

$f_2(z) = \sin z = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

$f_3(z) = \cos z \rightarrow \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y.$

$f_4 = e^z = e^{x+iy}$

$= e^x e^{iy}$ Note= $\sin it = +i \sinh t$

$\cos it = \cosh t$

$\cancel{f_5(z)} = e^x \cos y + i e^x \sin y.$

$f_5(z) = \alpha_i f_i(z) + \beta f_j(z).$

$f_6(z) = f_i(z) \cdot f_j(z).$

$f_7(z) = \frac{f_i(z)}{e^{f_i(z)}}$

$f_8(z) = f_i \circ f_j(z).$

Note: Each function our list has domain $\mathbb{C}$.

If a function our list is <u>never zero</u> at any point in $\mathbb{C}$ $\Leftrightarrow$ $e^{f_i(z)}$

B. class function :-

$$H(z) = \frac{f_i(z)}{f_j(z)}$$

Domain of $H(z) = \mathbb{C} - Z(f_i(z))$

$Z(f_i) = \{ t \in \mathbb{C} : f_j(t) = 0 \}$

Important

C-

$K(z) = f_i(H(z)) = f_i \circ H(z)$

Dod of $K(z)$ = Dod of $H(z)$.

Example:- $\sin \frac{1}{z}$, $e^{\frac{1}{z}}$, $e^{\frac{\sin z}{z}}$.

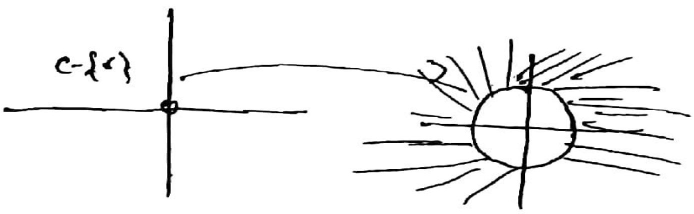
Example !- $f(z) = f(re^{i\theta}) = (r+1)e^{i\theta}$

$f : A \longrightarrow B$.

$A = \mathbb{C} - \{0\}$, $B = \{ z \in \mathbb{C} : |z| > 1 \}$.

$f(z) = re^{i\theta} + 1 \cdot e^{i\theta}$

$= z + e^{i\theta} = z + \frac{z}{|z|}$



$\Rightarrow$ f is one-one and onto from A to B

$g(re^{i\theta}) = g(z) = (r+1)\, e^{i\theta}$

$g : B \longrightarrow A$

$\Rightarrow$ f and g both are invertible from A to B and B to A respectively.

If $f \in \mathbb{C}[z, \bar{z}]$

$\Rightarrow$ f is polynomial in two variable z and $\bar{z}$

$f(z) = f(z, \bar{z}) = P_1(x, y) + P_2(x, y)\, i$

$P_1, P_2 \in \mathbb{R}[x, y]$

Example:-

$f(z) = |z|^2 + 2z + i\bar{z}$

$= z\bar{z} + 2z + i\bar{z}$

$= x^2+y^2+2(x+iy) + ix + y$

$= (x^2+y^2+2x+y) + i(2y+x)$

Example:-

$f(z) = x^2 - y^2 + x + xyi$

$= \left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 + \frac{z+\bar{z}}{2} +$

$\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right) i \ldots$

\# if $w = f(z) = u + iv$

$\Rightarrow \bar{w} = \overline{f(z)} = u - iv$

and $\boxed{f(\bar{z}) = u(x, -y) + i\, v(x, -y).}$

Example:- $f(z) = 2z^2 + iz + 2 + 3i$

$\overline{f(z)} = \bar{z}^2 - i\bar{z} + 2 - 3i$

$f(\bar{z}) = 2\bar{z}^2 + i\bar{z} + 2 + 3i$

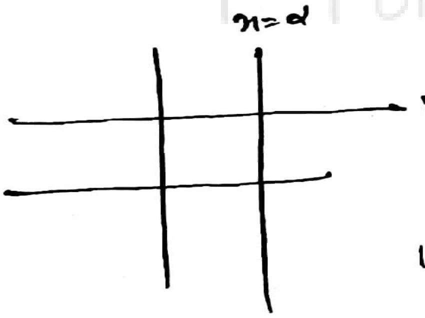
Bounded function:- Let f is define on $D \subseteq \mathbb{C}$. We say f is bounded on in D if $f(D)$ is bounded subset of $\mathbb{C}$.

i.e. f is bounded on D if $\exists\, M \in \mathbb{R}^+$ such that $\forall z \in D$.

$|f(z)| \leq M$.

Example:- $f(z) = \sin z$

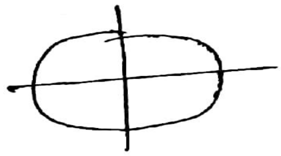
$= \sin x \cosh y + i \cos x \sinh y$



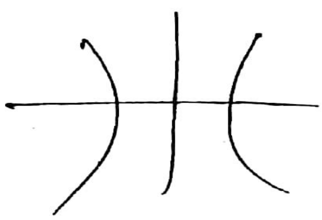
$u = \sin x \cosh y. \Rightarrow u = a \sin x$

$v = \cos x \sinh y \Rightarrow v = b \cos x$

$$\boxed{\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1}$$



$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \rightarrow$ Hyperbola

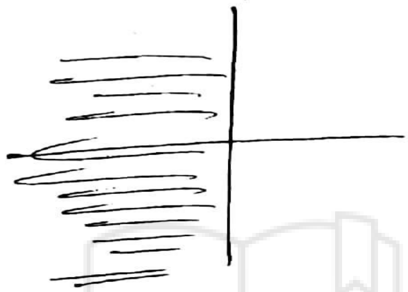


Example:

$$f(z) = e^{z}$$

$$|f(z)| = |e^{x+iy}| = e^{x} < 1, \quad x < 0$$

$$\Rightarrow \; |f(z)| < 1 \quad C: \operatorname{Re} z < 0, \; \forall z \in \mathbb{C}$$

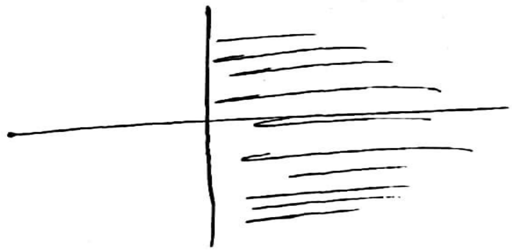


Example:

$$f(z) = e^{-z} = e^{-x-iy}$$

$$|f(z)| = |e^{-x} e^{-iy}| \le e^{-x}$$

$|f(z)| < 1 \; : \forall z \in \mathbb{C} \; :$ Re part $z > 0$



Example: $f(z) = e^{-z^4}$

$f(z) = e^{-y^4} < 1$ on y-axis

$f(z) = e^{-x^4}$ on x-axis