



# THE IIT - JEE SECRET

JEE MAINS AND JEE ADVANCED

MATHEMATICS - III

Calculus - I



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# FUNCTION

Cartesian product of 2 sets ( $A \times B$ )

$$A \equiv \{a, b, c, d\} \quad A \times B \equiv \{(a, b) \mid a \in A, b \in B\}$$

$$B \equiv \{p, q\}$$

$$A \times B = \{(a, p), (a, q), \dots, (d, p), (d, q)\}$$

$$n(A) = m \quad n(B) = n$$

$$n(A \times B) = mn$$

If  $n(A) = m, n(B) = n$   
 Total no. of subsets of  $A \times B = 2^{mn}$

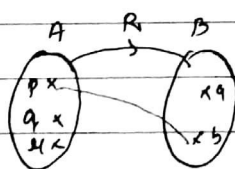
Relation

$$A \equiv \{p, q, r\}$$

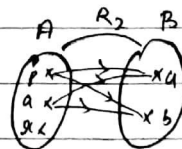
$$B \equiv \{a, b\}$$

It is defined as from set A to set B is any subset of  $A \times B$

$$R \subset A \times B \quad ; \quad R \equiv \{(a, b) \mid a \in A, b \in B\}$$



$$R_1 \equiv \{(p, b)\}$$

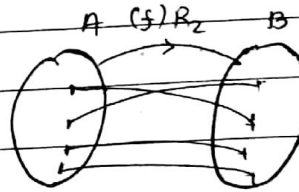
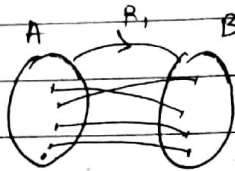


$$R_2 \equiv \{(p, a), (p, b), (q, a), (q, b)\}$$

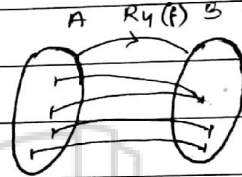
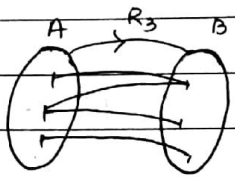
Function

△ relation from set A to set B is called  
 if every element of set A is connected to some  
unique element of set B, it is denoted as

$$f: A \rightarrow B$$



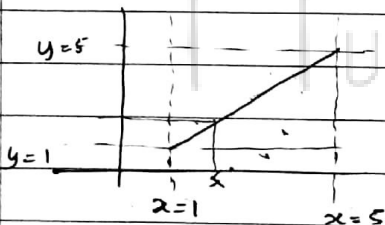
$$f: A \rightarrow B$$



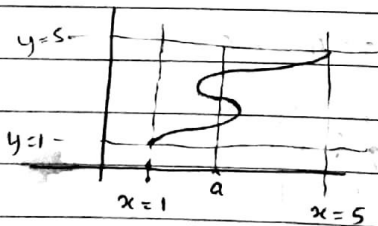
$$f: A \rightarrow B$$

Identify which of the following from  $A$  to  $B$  represents function.

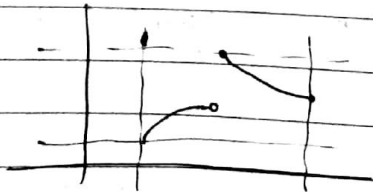
$$A = [1, 5] \quad B = [1, 5]$$



It is a function



Not a function

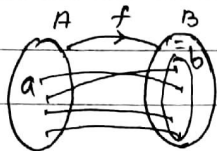


It is a function

Notes:-

If a vertical line cuts the given graph in more than 1 point then it can't be graph of function.

$$f: A \rightarrow B$$



$A \rightarrow$  domain of the  $f^n$

$B \rightarrow$  co-domain of the  $f^n$

$$b = f(a)$$

$b$  is image of  $a$  under  $f$

$a$  is pre-image of  $b$  under  $f$

Range of  $f^n$

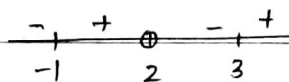
The set of all  $f$ -images is called range of  $f^n$

$$\text{range} \subseteq \text{co-domain}$$

Find the domains of function :-

$$f(x) = \sqrt{\frac{(x+1)(x-3)}{(x-2)}}$$

$$\frac{(x+1)(x-3)}{(x-2)} \geq 0$$



$$D_f = [-1, 2) \cup [3, \infty)$$

$$f(x) = \log_{(7-x)}(x-5)$$

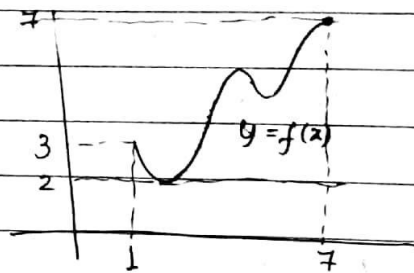
$$7-x > 0 \text{ and } 7-x \neq 1 \text{ and } x-5 > 0$$

$$x < 7$$

$$x \neq 6$$

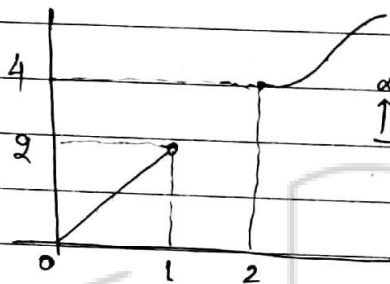
$$x > 5$$

$$D_f = (5, 6) \cup (6, 7)$$



$$D_f \in [1, 7]$$

$$R_f \in [2, 7]$$



$$f: [0, 1) \cup [2, \infty) \rightarrow \mathbb{R}$$

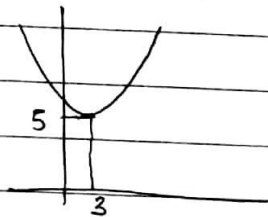
$$R_f [0, 2) \cup [4, \infty)$$

Find the range of following functions:-

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = x^2 - 6x + 14$$

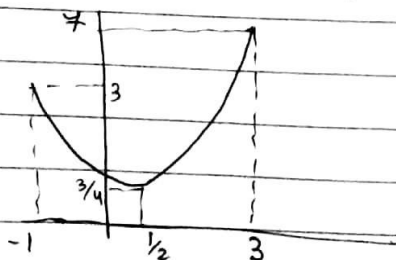
$$f(x) = [5, \infty) \quad \left( \frac{-b}{2a}, \frac{-D}{4a} \right)$$



$$f: [-1, 3] \rightarrow \mathbb{R}$$

$$f(x) = x^2 - x + 1$$

$$\text{vertex } \left( \frac{1}{2}, \frac{3}{4} \right)$$



$$R_f \left[ \frac{3}{4}, 7 \right]$$

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = 8 - 3\sin x$$

$$-1 \leq \sin x \leq 1$$

$$-3 \leq 3\sin x \leq 3$$

$$5 \leq 8 - 3\sin x \leq 11$$

$$f(x) = \frac{1}{\sqrt{9 + 5\sin x}}$$

$$-5 \leq \sin x \leq 5$$

$$4 \leq 9 + 5\sin x \leq 14$$

$$2 \leq \sqrt{9 + 5\sin x} \leq \sqrt{14}$$

$$\frac{1}{\sqrt{14}} \leq \frac{1}{\sqrt{9 + 5\sin x}} \leq \frac{1}{2}$$

$$\left[ \frac{1}{\sqrt{14}}, \frac{1}{2} \right]$$

Note:-

$$-2 < x < 5$$

$$-2 < x < 0 \quad \text{or} \quad 0 < x < 5$$

$$-\infty < \frac{1}{x} < -\frac{1}{2} \quad \text{or} \quad \frac{1}{5} < \frac{1}{x} < \infty$$

## Some Important Functions

→ Polynomial Functions

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n$$

$$a_0 \neq 0, \quad n \in \mathbb{N}$$

'f' is a polynomial of degree 'n'

Some notes

The graph of poly. f<sup>n</sup> is always continuous.

The range of odd degree polynomial is  $\mathbb{R}(-\infty, \infty)$

$$f(x) = x^7 - 5x^4 + x^2 + x + 1$$

$$= x^7 \left( 1 - \frac{5}{x^3} + \frac{1}{x^5} + \frac{1}{x^6} + \frac{1}{x^7} \right)$$

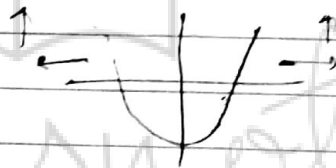


$$x \rightarrow \infty \quad f \rightarrow \infty$$

$$x \rightarrow -\infty \quad f \rightarrow -\infty$$

The range of even degree polynomial can't be  $\mathbb{R}$

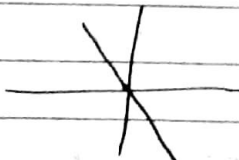
$$f(x) = 2x^6 - x^5 + 3x^2 + x + 1$$



The function of form  $f(x) = ax$ ,  $a \neq 0$ , can not be or is called odd lined polynomial.



The function of form  $f(x) = ax + b$ ,  $a \neq 0$  is called linear polynomial.



## Algebraic Functions

A function  $f$  is called algebraic function if it can be constructed using algebraic operations such as  $+$ ,  $-$ ,  $\times$ ,  $\div$  and using radical signs.

Examples:-

$$f = x^2 - 2x + 3$$

$$f = \sqrt{x}$$

$$f(x) = |x|$$

$$f(x) = \sqrt{x^2}$$

$$f(x) = \frac{x^2 + x^3 + x - 3x^{3/2}}{(\sqrt{x^2 + x + 1}) \cdot 2x^{-1}}$$

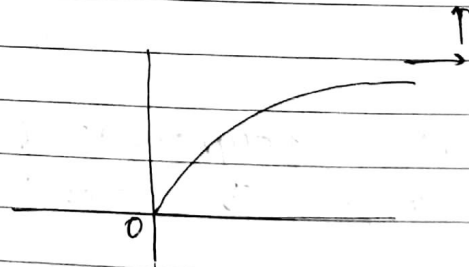
All polynomial functions are algebraic but not converse.

Functions which are not algebraic are called transcendental functions.

$$f(x) = \sqrt{x}$$

$$D_f \rightarrow \mathbb{R}^+ \cup \{0\}$$

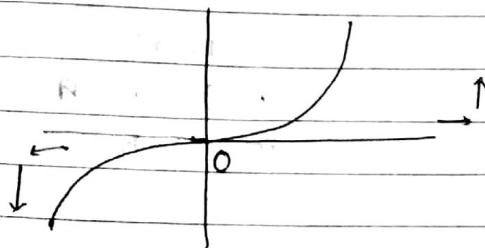
$$R_f \rightarrow [0, \infty)$$



$$f(x) = x^3$$

$$R_f \rightarrow (-\infty, \infty)$$

$$D_f \rightarrow \mathbb{R}$$



## Rational Functions

The functions of the form  $f(x) = \frac{g(x)}{h(x)}$  where  $g(x)$  &  $h(x)$  are polynomial functions are called rational functions.

Examples -

$$f(x) = \frac{x-1}{x-2}$$

$$f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}$$

$$f(x) = \frac{1}{x} \quad D_f \rightarrow \mathbb{R} - \{0\}$$

$$y = \frac{1}{x} \quad x = \frac{1}{y} \quad \mathbb{R} - \{0\} \rightarrow \text{range}$$

## Exponential Functions

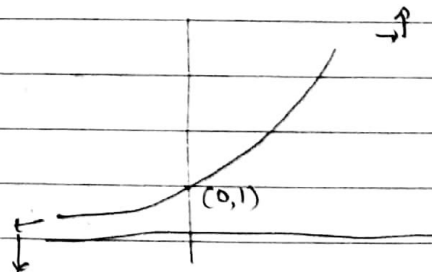
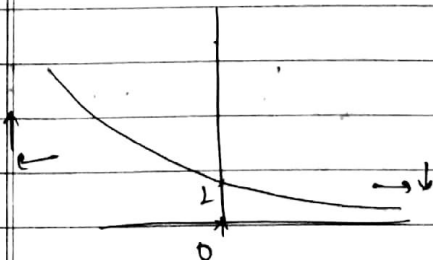
$$f(x) = a^x \quad a \neq 1 \text{ and } a > 0$$

$$0 < a < 1$$

$$f(x) = a^x$$

$$a > 1$$

$$f(x) = a^x$$



$$D_f \rightarrow \mathbb{R}$$

$$R_f \rightarrow \mathbb{R}^+$$

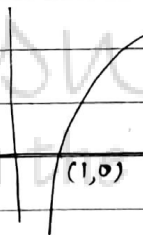
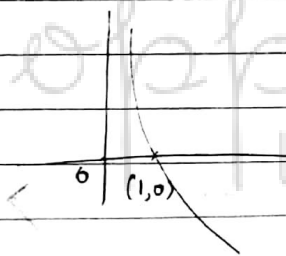
|                    | $R_f$         | $D_f$             |
|--------------------|---------------|-------------------|
| $y = 2^{x^3} + 1$  | $(1, \infty)$ | $R$               |
| $y = 2^{\tan x}$   | $R^+$         | $R - (2n-1)\pi/2$ |
| $y = 2^{\sin x}$   | $[1/2, 2]$    | $R$               |
| $f(x) = e^{\ln x}$ | $R^+$         | $R^+$             |
| $= x$              |               |                   |

## Logarithm Functions

$$f(x) = \log_a x \quad D_f \rightarrow R^+$$

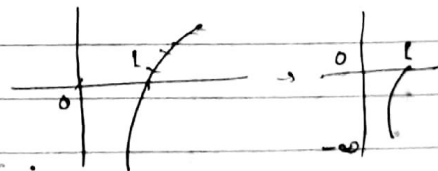
$$0 < a < 1$$

$$a > 1$$



| $f_n$                  | $D_f$       | $R_f$          |
|------------------------|-------------|----------------|
| $f(x) = \log x^2$      | $R - \{0\}$ | $R$            |
| $f(x) = \log  \sin x $ | $R - n\pi$  | $(-\infty, 0]$ |

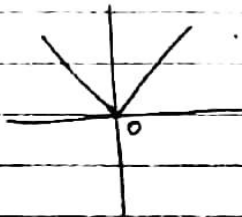
$$\begin{array}{c}
 \log(\quad) \\
 \uparrow \\
 (0, 1] \\
 |\sin x| \\
 \uparrow \\
 x
 \end{array}$$



## Absolute Value Function [Modulus Function]

$$f(x) = |x| = \begin{cases} -x & , x < 0 \\ x & , x \geq 0 \end{cases}$$

$$f(x) = \begin{cases} -x & , x < 0 \\ x & , x \geq 0 \end{cases} \quad \text{non-uniform f.}$$



$$D_f \rightarrow \mathbb{R}$$

$$R_f \rightarrow [0, \infty)$$

$$f(x) = \frac{1}{|x|} \quad D_f \rightarrow \mathbb{R} - \{0\}$$

$$y = \frac{1}{|x|}$$

$$|x| = \frac{1}{y}$$

$$\frac{1}{y} > 0$$

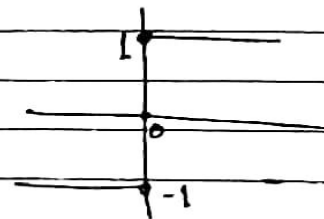
$$R_f \rightarrow \mathbb{R}^+$$

## Signum Functions

$$f(x) = \text{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

$$D_f \rightarrow \mathbb{R}$$

$$R_f = \{-1, 0, 1\}$$



$$f(x) = \text{sgn}(x^2 + x + 1)$$

$$D_f \rightarrow \mathbb{R}, \quad R_f \rightarrow \{-1, 0, 1\}$$

## Greatest Integer Functions

$$f(x) = [x]$$

$$x = I + f \quad I \rightarrow \text{int.}$$

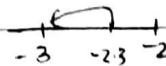
$$[x] = I$$

$$0 \leq f < 1$$

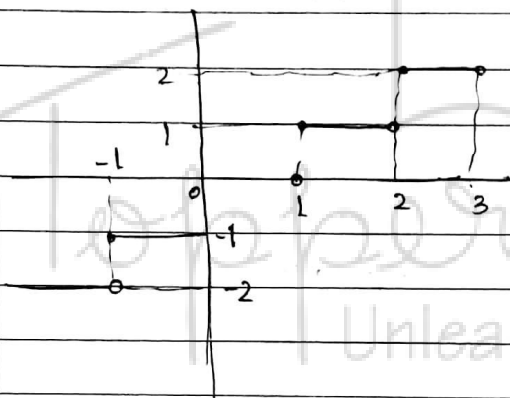
$$[1.9999] = 1$$



$$[-2.3] = -3$$



$$f(x) = [x] = \begin{cases} -1 & -1 \leq x < 0 \\ 0 & 0 \leq x < 1 \\ 1 & 1 \leq x < 2 \end{cases}$$



step-up function

$$D_f \rightarrow I$$

$$[x] = 2$$

$$2 \leq x < 3$$

$$[2, 3)$$

$$2[x] = 3$$

N.S.

$$[x] = 0$$

$$0 \leq x < 1$$

$$[x] = -2$$

$$-2 \leq x < -1$$

$$[x] \leq 4$$

$$x < 4$$

$$(-\infty, 4)$$

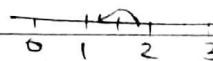
$$1 < [x] < 3$$

$$2 \leq x < 3$$

$$[x] \leq 4$$

$$-\infty < x < 5$$

$$(-\infty, 5)$$



## Properties of GZF

\*(a)  $[x+m] = [x] + m ; m \in \mathbb{I}$

$\downarrow$   
 $\mathbb{I} + \mathbb{I}$

\*(b)  $[x] + [-x] = \begin{cases} 0 & ; x \in \mathbb{I} \\ -1 & ; x \notin \mathbb{I} \end{cases}$

$\downarrow$   
 $\mathbb{I} + \mathbb{I}$   
 $\mathbb{I}$

$\downarrow$   
 $-\mathbb{I} - \mathbb{I}$   
 $-1 - \mathbb{I} + 1 - \mathbb{I}$   
 $-1 - \mathbb{I}$

(c)  $[x] \leq x < [x] + 1$

$\mathbb{I} \leq \mathbb{I} + \mathbb{I} \leq \mathbb{I} + 1$

(d)  $[x] + [y] \leq [x+y] \leq [x] + [y] + 1$

$\mathbb{I}_1 + \mathbb{I}_2 + f_1 + f_2$

$0 \leq f_1 < 1$

$0 \leq f_2 < 1$

$0 \leq f_1 + f_2 < 2$

Find the domain and range of  $f$  if  $f(x) = \frac{1}{[x]}$

$f(x) = \frac{1}{[x]}$

$[x] = 0$

$0 \leq x < 1$

$[0, 1)$

$D_f \rightarrow \mathbb{R} - [0, 1)$

$(-\infty, 0) \cup [1, \infty)$

$R_f = \left\{ \frac{1}{N} \mid N \in \mathbb{I}, N \neq 0 \right\}$